

On the extended Kolmogorov-Nagumo information-entropy theory, the $q \rightarrow \frac{1}{q}$ duality and its possible implications for a non-extensive two dimensional Ising model.

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Abstract

The aim of this paper is to investigate the $q \rightarrow \frac{1}{q}$ duality in an information-entropy theory of all q -generalized entropy functionals (Tsallis, Renyi and Sharma-Mittal measures) in the light of a representation based on generalized exponential and logarithm functions subjected to Kolmogorov's and Nagumo's averaging. We show that it is precisely in this representation that the form invariance of all entropy functionals is maintained under the action of this duality. The generalized partition function also results to be a scalar invariant under the $q \rightarrow \frac{1}{q}$ transformation which can be interpreted as a non-extensive two dimensional Ising model duality between systems governed by two different power law long-range interactions and temperatures. This does not hold only for Tsallis statistics, but is a characteristic feature of all stationary distributions described by q -exponential Boltzmann factors.

Keywords: Generalized information entropy measures, Tsallis, Renyi, Sharma-Mittal, Maximum entropy principle, Ising duality.

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1 Introduction

Because of a growing interest in non-extensive thermostatics, during recent years, a good deal of study has been devoted to a non-extensive generalization of Boltzmann-Gibbs statistical mechanics.[1] A generalization of the ordinary formalism was however mainly devoted to the study of C. Tsallis' entropy measure [2] and separately to a lesser degree to Renyi's entropy functional [3], both indexed by a q -parameter. Tsallis measure has been preferred over other q -generalized measures since the stability and concavity properties of Renyi's entropy are still a matter of debate. Therefore only few efforts have been pursued to unite them in a common and natural theoretical framework.

Nevertheless some attempts in this direction have been made, e.g. by Frank and Plastino [4] and Frank and Daffertshofer [5]. In a previous communication [6], a more analytic approach was used using a Kolmogorov-Nagumo generalized logarithmic-exponential representation (KNG-representation). We could write down the set of entropy functionals, from Boltzmann-Gibbs (BG) entropy through Renyi and Tsallis, up to Sharma-Mittal (SM) [7] and a new type of entropy measure we called "supra-extensive". Here the increasing generalization of entropy measures from arithmetic to non-arithmetic averages, and from additive to non-additive systems became clear in its hierarchical structure (for similar attempts see also [8]). With this formalism it is also easier to highlight the relationship that exists between generalized information-entropy measures and Fisher information.[9] It is so far uncertain if this can have a wider application in thermostatics, but since this representation unites in one frame all pseudoadditive and KN-averaged measures leading to maximum entropy probability distributions (MEPDs) (of entropies subjected to escort-means) exhibiting q -generalized Boltzmann factors, it is worthwhile to study its overall structure.

In this paper we will investigate this deformed logarithm-exponential representation focussing our attention on the dual description of generalized statistics. We show how in this representation a deeper meaning of the $q \rightarrow \frac{1}{q}$ duality emerges, first studying to what kind of formal invariance it leads and then highlighting in what way one can relate it to a non-extensive Ising model duality.

2 Overview of the generalizing path for entropy measures

We shortly recall how the generalization of Tsallis and Renyi measures leads to Sharma-Mittal information describing its formal structure in terms of KN-averages (for some further details see [4], [6], [8]).

2.1 How to construct generalized KN-averaged information-entropy measures

The point of departure is the well known BG entropy

$$S_{BG}(P) = -k \sum_i p_i \log p_i, \quad (1)$$

with $P = \{p_1, \dots, p_\Omega\}$ ($i = 1, \dots, \Omega$) a probability distribution (PD) with p_i the probability of the system to be in the i -th microstate, k the Boltzmann constant and Ω the total number of accessible microstates. From now on we will use Shannon's information ($k = 1$) and write it

$$S_S(P) = \sum_i p_i \log \left(\frac{1}{p_i} \right). \quad (2)$$

Given two PD's A and B , BG and Shannon's entropies follow an additive law

$$S_S(A \cap B) = S_S(A) + S_S(B|A), \quad (3)$$

with $S_S(B|A) = \sum_i p_i(A) S_S(B|A_i)$ the conditional information of B given A. Shannon's measure is also recovered if one imposes axiomatically additivity through Shannon-Khinchin's axioms [19].

However, additivity and/or extensivity can be violated in systems governed by nonlinear dynamics, with long range interaction forces, with fractal structure or for other "non-ordinary" deterministic types of behavior. According to a research trend that attempts to generalize thermostatics, these systems (but not all, see e.g. [10]) can be better described by a generalization of the BG-Shannon measure. This

was first introduced by Havrda and Charvat [11] in the information theory frame, and later by Tsallis for a generalized thermostatics [2]:

$$S_T(P, q) = \frac{\sum_i p_i^q - 1}{1 - q} = \frac{1}{q - 1} \sum_i p_i (1 - p_i^{q-1}), \quad (4)$$

with $q > 0$ a real parameter and where for $q \rightarrow 1$ one recovers Shannon's measure. It can be written also as a *q-deformed Shannon entropy*

$$S_T(P, q) = - \sum_i p_i^q \log_q p_i = \sum_i p_i \log_q \left(\frac{1}{p_i} \right), \quad (5)$$

where the *generalized q-logarithm function*

$$\log_q x = \frac{x^{1-q} - 1}{1 - q}, \quad (6)$$

has been introduced, and which, for $q = 1$, becomes the common natural logarithm.

Having introduced power law scaled PD's, Beck and Schlögl defined [12] what they called *escort-distributions*

$$\pi_i(P, q) = \frac{p_i^q}{\sum_j p_j^q}; \quad q > 0, \quad q \in \mathbb{R}. \quad (7)$$

Accordingly one defines as *normalized q-average* or *escort average* of an observable \mathcal{O}

$$\langle \mathcal{O} \rangle^P = \sum_i \pi_i(P, q) \mathcal{O}_i. \quad (8)$$

Tsallis entropy is *pseudo-additive*

$$S_T(A \cap B) = S_T(A) + S_T(B|A) + (1 - q)S_T(A)S_T(B|A). \quad (9)$$

where

$$S_T(B|A) = \sum_i \pi_i(A) S_T(B|A = A_i), \quad (10)$$

with $\pi_i(A)$ defined in (7).

In a similar fashion as we have seen for (3), the generalization to conditional entropies is possible too. S. Abe [13] showed that the Shannon-Khinchin axioms can be generalized to Tsallis' entropy introducing what he called the *non-extensive conditional entropy* $S_T(B|A)$.

Writing entropies with the inverse of the PD, like in the r.h.s. of (2) and (5), highlights its structure better since in a generalized context $\log_s x^\alpha \neq \alpha \log_s x$ if $s \neq 1$. Then one sees that an information measure is an average of the *elementary information gains*

$$I_i \equiv I_i \left(\frac{1}{p_i} \right) = \log_s \left(\frac{1}{p_i} \right) \quad (11)$$

associated to the i-th event of probability p_i , which for Shannon's and Tsallis' entropy means

$$S_S(P) = \left\langle \log \left(\frac{1}{p_i} \right) \right\rangle_{lin} \quad (12)$$

and

$$S_T(P) = \left\langle \log_q \left(\frac{1}{p_i} \right) \right\rangle_{lin} \quad (13)$$

respectively, with the common *arithmetic*-, or *linear mean* $S = \sum_i p_i I_i$.

A.N. Kolmogorov and M. Nagumo ([14], [15]) realized independently from each other that, if Kolmogorov's axioms are supposed to be the foundation of probability theory, then the notion of average can be generalized in more abstract terms to what is called a *quasi-arithmetic* or *quasi-linear mean*. It turns out to be

$$S = f^{-1} \left(\sum_i p_i f(I_i) \right), \quad (14)$$

with f a strictly monotonic continuous function defined on real numbers I_i , called the *Kolmogorov-Nagumo function* (KN-function) and p_i ($i = 1, \dots, \Omega$) non-negative weighting constants such that $\sum_i p_i = 1$.¹

Later Renyi showed [3] that if one imposes additivity on an entropy measure then there are only two possible KN-functions left. The first leads to the conventional linear mean with KN-function

$$f(x) = x, \quad (15)$$

and the second is the *exponential mean* represented by KN-function

$$f(x) = c_1 b^{(1-q)x} + c_2; \quad q \in \mathbb{R} \quad (16)$$

with c_1 and c_2 two arbitrary constants and b the logarithm base (we set $b=e$).

Shannon entropy comes from the former KN-function averaging over the elementary information gains $I_i = \log(1/p_i)$. *Renyi's information measure* uses the latter one but averages the same information gains “exponentially” and writes

$$S_R(P, q) = \frac{1}{1-q} \log \sum_i p_i^q. \quad (17)$$

For $q \rightarrow 1$ it becomes Shannon entropy. Renyi in this sense extended information measures but maintained it to be additive.

Now, $S_R(P, q)$ is invariant under the choice of c_1 and c_2 . Therefore we don't loose in generality if in (16) one sets $c_1 = \frac{1}{1-q} = -c_2$, which, because of (6), is equivalent to

$$f(x) = \log_q e^x, \quad (18)$$

and inserted in (14) with

$$I_i = \log \left(\frac{1}{p_i} \right), \quad (19)$$

shows that (17) is

$$S_R(P, q) = \left\langle \log \left(\frac{1}{p_i} \right) \right\rangle_{\text{exp}}, \quad (20)$$

with $\langle \cdot \rangle_{\text{exp}}$ an average defined by KN-function (18). Compare this with (12) and (13). One might call Renyi's entropy also an “exponentially averaged Shannon entropy”. Napier's logarithm assures additivity while (18) imposes the exponential average.

It should be noted how P. Jizba and T. Arimitsu [18] showed that Renyi's measure can be obtained also from the extension of the Shannon-Khinchin axioms [19] to a quasi-linear conditional information

$$S_R(B|A) = f^{-1} \left(\sum_i \pi_i(A) f(S_R(B|A_i)) \right), \quad (21)$$

with f as given in (16).

One might ask if there exists a measure which is non-additive and non-linearly averaged but contains Tsallis' and Renyi's entropies as special cases. There are two ways to do this. The first by working with the generalized exponentials and logarithms, the other through an extension of the KN-mean.

Note that Tsallis and Renyi entropies have a special relationship:

$$S_R(P, q) = \frac{1}{1-q} \log [1 + (1-q) S_T(P, q)]. \quad (22)$$

By using generalized logarithms (6) and the generalized exponential

$$e_q^x \equiv e_q[x] = [1 + (1-q)x]^{\frac{1}{1-q}}, \quad (23)$$

¹Typical examples are: $f(x) = x$ the weighted mean (arithmetic mean if every $p_i = 1/\Omega$), $f(x) = x^2$ the root mean square, $f(x) = \log x$ the geometric mean, etc. Kolmogorov and Nagumo considered only the case of the arithmetic means (equal weights), while the generalization to arbitrary weights as given in (14) was introduced by B. de Finetti [16] and T. Kitagawa [17] and others later.

(which becomes the exponential function for $q = 1$) this becomes

$$S_R(P, q) = \log e_q^{S_T(P, q)}, \quad (24)$$

and obviously

$$S_T(P, q) = \log_q e^{S_R(P, q)}. \quad (25)$$

Eq. (24) suggest immediately a further two parametric generalization:

$$S_{SM}(P, \{q, r\}) = \log_r e_q^{S_T(P, q)} = \frac{1}{1-r} \left[\left(\sum_i p_i^q \right)^{\frac{1-r}{1-q}} - 1 \right], \quad (26)$$

with r another real parameter.²

Equation (26) turns out to be B.D. Sharma and D.P. Mittal's information measure [7]. They used it in the frame of information theory about thirty years ago and it was more recently investigated in the context of a generalized thermostatics by Frank, Daffertshofer and Plastino ([4], [5]).

If one remembers how the q -logarithm satisfies the pseudo-additive law

$$\log_q xy = \log_q x + \log_q y + (1-q)(\log_q x)(\log_q y) \quad (27)$$

and notes the decomposition

$$e_q^{x+y+(1-q)xy} = e_q^x e_q^y, \quad (28)$$

then, in this representation, it is immediately visible that a pseudo-additive law is satisfied also for SM entropy (start from (26), use (9))

$$S_{SM}(A \cap B) = \log_r e_q^{S_T(A \cap B)} = \log_r (e_q^{S_T(A) + S_T(B|A) + (1-q)S_T(A)S_T(B|A)}) = \quad (29)$$

$$= \log_r (e_q^{S_T(A)} e_q^{S_T(B|A)}) = S_{SM}(A) + S_{SM}(B|A) + (1-r)S_{SM}(A)S_{SM}(B|A), \quad (30)$$

with A and B the families of PDs of two statistical systems.

Note that it is the magnitude of parameter r which stands for the degree of pseudo-additivity, while q stands for a PD deformation parameter. In Tsallis' entropy, when $r \rightarrow q$, the pseudo-additive parameter r merges into the deformation parameter q , and outside the frame of a KNG-representation, the two appear somehow mixed together.

But, and this is the equivalent second method, SM entropy emerges also by q -generalizing in the most simple possible way the KN-mean (18) to

$$f(x) = \log_q e_r^x, \quad (31)$$

which we call the *quasi-exponential mean*, and (26) can be written as

$$S_{SM}(P, \{q, r\}) = \left\langle \log_r \left(\frac{1}{p_i} \right) \right\rangle_{q-\exp}. \quad (32)$$

2.2 The generalized information-entropy measures hierarchy

Manipulating with the formalism, frequently it turns out useful to rewrite in different alternative ways the quantity

$$\left(\sum_i p_i^q \right)^{\frac{1}{1-q}} = \left\langle \left(\frac{1}{p_i} \right)^{1-q} \right\rangle_{\text{lin}} = e_q^{\left\langle \log_q \left(\frac{1}{p_i} \right) \right\rangle_{\text{lin}}} = e_q^{S_T(P, q)} = e^{S_R(P, q)}. \quad (33)$$

But writing it in the KNG-representation leads also to the following equality

$$\Gamma(P, q) \equiv \left\langle \frac{1}{p_i} \right\rangle_{\log_q} = \left(\sum_i p_i^q \right)^{\frac{1}{1-q}}, \quad (34)$$

²From (25) one naturally obtains also another measure: $S_{SE}(P, \{q, r\}) = \log_q e_r^{S_R(P, q)} = \frac{\left[1 + \frac{(1-r)}{(1-q)} \log \sum_i p_i^q \right]^{\frac{1-q}{1-r}} - 1}{1-q}$. We called it "supra-extensive" in [6]. It does however no longer follow a KN averaging theory and therefore we will not consider it here.

where we used what we call the *generalized logarithmic mean* $\langle \cdot \rangle_{\log_q}$ defined by the KN-function $f(x) = \log_q x$.

Then, entropies (24), (25) and (26), using (33) in form of (34), can be written in an overall generalized hierarchy of information measures as follows

$$S_{SM}(P, \{q, r\}) = \log_r \Gamma(P, q) = \log_r \left\langle \frac{1}{p_i} \right\rangle_{\log_q} ; \quad (35)$$

$$S_T(P, q) = \log_q \Gamma(P, q) = \log_q \left\langle \frac{1}{p_i} \right\rangle_{\log_q} ; \quad (36)$$

$$S_R(P, q) = \log \Gamma(P, q) = \log \left\langle \frac{1}{p_i} \right\rangle_{\log_q} ; \quad (37)$$

$$S_S(P) = \log \Gamma(P, 1) = \log \left\langle \frac{1}{p_i} \right\rangle_{\log} , \quad (38)$$

where also we added Shannon's measure.

Then one sees how SM's entropy generalizes Renyi's additive entropy to pseudo-additivity, characterized by the r -logarithm. This formalism makes it also easier to recognize the behavior of the limits for $r \rightarrow q$ or $r \rightarrow 1$ of (35) than in their explicit form (the r.h.s. of (26)). Looking also at its " $\log_r \times \exp_q$ form" (the middle term of (26)) it is immediately visible (without any need to apply Hopital rule, first order approximations or whatever) how, for $r \rightarrow q$ and $r \rightarrow 1$, they reduce to Tsallis and Renyi entropy respectively.

According to ordinary extensive statistical mechanics entropy can be defined as $S = k \log \Gamma$, with Γ the entire phase-space volume of the system (Γ -space). In the KNG-representation, from (38) (and unitary Boltzmann constant) this is given by $\Gamma(P, 1) = \log \left\langle \frac{1}{p_i} \right\rangle_{\log} = e^{S_S(P, 1)} = e^{\sum_i p_i \log \frac{1}{p_i}}$. While in general, the above information-entropy measures hierarchy is highlighted in terms of KN q -generalized Γ -space volumes $\Gamma(P, q) = \left\langle \frac{1}{p_i} \right\rangle_{\log_q} = (\sum_i p_i^q)^{\frac{1}{1-q}}$.

2.3 Some remarks on the generalizing path taken

One can see [4] that MEPDs (which from now on will be labeled with a caret symbol $\hat{\cdot}$) obtained from SM entropy subjected to the escort Hamiltonian

$$\langle H \rangle = \frac{\sum_i p_i^q E_i}{\sum_i p_i^q} = U_g \quad (39)$$

are of the form

$$\hat{p}_i^{SM} = \frac{e_q[-\beta_g(E_i - U_g)]}{Z_{SM}} = \frac{e_q[-\bar{\beta}_g E_i]}{\bar{Z}_{SM}} \quad (40)$$

with

$$Z_{SM} = \sum_j e_q[-\beta_g(E_j - U_g)] ; \quad \bar{Z}_{SM} = \sum_j e_q[-\bar{\beta}_g E_j] = \frac{Z_{SM}}{e_q[\beta_g U_g]} \quad (41)$$

the generalized partition function, β_g the generalized Lagrangian parameter (where the parameter r has been absorbed) and E_j the microstate energy.³ The terms on the r.h.s. of (40) and (41) highlight the mean energy translational invariance of the MEPDs with $\bar{\beta}_g = \frac{\beta_g}{1+(1-q)\beta_g U}$.

It is therefore important to remember that a q -generalized Boltzmann factor does not necessarily indicate that one is dealing with Tsallis statistics. It is a more general feature which is common to Renyi and SM entropic functionals too. It is instead β_g (or $\bar{\beta}_g$) which selects one or the other statistics. In this regards let us shortly comment on an issue which needs some clarification when generalized entropies other than that of Tsallis are considered in a generalized thermostatics.

It should be mentioned that there are some reasons to argue that the Renyi entropy is not Lesche-stable (for $q \neq 1$), is concave only for $q \leq 1$, and does not possess the property of finite entropy production per

³Note that in general eq. 40 does not maintain itself always positive and real without cut-off prescriptions. More on this can be found in [20] and references therein.

unit time. And one should expect that further generalizations of it, like SM entropy, cannot possess these properties either.⁴ If this has some thermodynamical implications is currently a controversial matter. There have been different types of critiques to Lesche's and Abe's approach (see e.g. some papers listed in [21]). It seems to us that this is still an open issue that so far has not been clarified entirely.

Moreover, the kind of statistics which emerges here is two parametric. While so far no evidence exists that natural systems are governed by more than a one parameter extended BG statistics, we will try to put forward some arguments which indicate that an extra parameter might furnish some new insights on the difficulties related to the contrasting physical interpretations and microscopic justification of the q -parameter (e.g. [22], [10], [23]). And fact is also that from an analytic point of view, as Frank and Plastino showed [4], neither Renyi nor Tsallis entropy are the most general pseudo-additive measures which allow for escort averages admitting of a partition function, but SM's measure is, which is two parametric. SM entropy is also consistent with the familiar q -invariant Legendre transform structure of thermodynamics and the generalizations here considered emerge in any case as a natural consequence of Kolmogorov's axiomatics applied to power law PDs.

Anyway, even if not in the frame of an extended thermostatics, these measures still can have an application at least in other research fields of information theory, as cybernetics, control theory, signal processing, cryptography or in a large class of chaotic dynamical system or in other areas of mathematical statistics.

3 The meaning of the $q \rightarrow \frac{1}{q}$ duality in the frame of the KNG-representation and some possible consequences

The notion of duality is used in rather different connections. Generally speaking, two physical theories are said to be dual if there exists a map, a duality transformation, so that the first theory can be transformed into a second theory looking just like the first one. A duality acts by interchanging the roles of two objects linking quantities that seemed to be separate (e.g. the electric with the magnetic fields, other strong and weak coupling strengths, large and small distance scales or, as in our case, two PDs which dually link the expectation values of an observable, like that of a Hamiltonian). These quantities have always been distinctive of some limits of a physical system in both classical and quantum field theories and it is then possible to observe the behavior of the observable in one model in terms of the other and eventually use perturbative and semiclassical methods.

Recent studies highlighted how the Tsallis entropy seems to lead just naturally to a dual description in terms of a $q \leftrightarrow \frac{1}{q}$ duality too. Its function consists in linking the distribution $P = \{p_i \dots p_\Omega\}$ with the escort distribution $\Pi = \{\pi_i \dots \pi_\Omega\}$ of (7), that is

$$\pi_i(P, q) = \frac{p_i^q}{\sum_j p_j^q} \quad \leftrightarrow \quad p_i(\Pi, q) = \frac{\pi_i^{1/q}}{\sum_j \pi_j^{1/q}}. \quad (42)$$

If one considers Π and P both normalized distributions it is easy to verify this by simply inserting one identity in the other.

This duality was first noted by Tsallis, Mendes and Plastino [24], and further investigated by Frank and Plastino [4], Naudts [25], and others (even earlier, Abe noted a similar duality [26], however of a quite different nature). We will call it the Tsallis, Mendes and Plastino duality, or simply the TMP-duality.

This dually relates also the q -expectation of an observable \mathcal{O} with its "classical" average as⁵

$$\langle \mathcal{O} \rangle^q = \frac{\sum_i p_i^q \mathcal{O}_i}{\sum_j p_j^q} \quad \Leftrightarrow \quad \frac{\sum_i \pi_i^{1/q} \mathcal{O}_i}{\sum_j \pi_j^{1/q}} = \langle \mathcal{O} \rangle^\pi \equiv \langle \mathcal{O} \rangle. \quad (43)$$

Here we will try to understand its deeper meaning and underline how the TMP-duality does not characterize only Tsallis' entropy but is a general transformation which works for every measure in a q -generalized statistics.

⁴But one should also be aware of the fact that, at least in principle, it still can recover it for a limited r -parameter space with $r \neq 1$.

⁵The fact that the TMP-duality maps an escort averaged observable into a linearly averaged one should not be confused with a transformation of a q -statistics into a BG one! The PD involved in this linear mean are here defined in the frame of a generalized statistics.

3.1 The TMP-duality as KNG-representation form invariant transformation

The TMP-duality has a particularly important effect in transforming the phase-space volume which becomes clear in the KNG-representation. Transformation (42) leads to the following identity

$$\Gamma(P, q) \equiv \left\langle \frac{1}{p_i} \right\rangle_{\log_q} = \left(\sum_i p_i^q \right)^{\frac{1}{1-q}} = \quad (44)$$

$$= \left(\sum_i \pi_i^{1/q} \right)^{\frac{-q}{1-q}} = \left\langle \frac{1}{\pi_i} \right\rangle_{\log_{1/q}} \equiv \Gamma(\Pi, \frac{1}{q}). \quad (45)$$

Simply insert the expression for p_i of (42) remembering the normalization condition and evaluate the logarithmic mean of $\frac{1}{\pi_i}$ with $q \rightarrow \frac{1}{q}$ according to (34).

Therefore the information-entropy hierarchy structure of (35) to (38) transforms into

$$S_{SM}(\Pi, \{q, r\}) = \log_r \Gamma(\Pi, \frac{1}{q}) = \log_r \left\langle \frac{1}{\pi_i} \right\rangle_{\log_{1/q}}; \quad (46)$$

$$S_T(\Pi, q) = \log_q \Gamma(\Pi, \frac{1}{q}) = \log_q \left\langle \frac{1}{\pi_i} \right\rangle_{\log_{1/q}}; \quad (47)$$

$$S_R(\Pi, q) = \log \Gamma(\Pi, \frac{1}{q}) = \log \left\langle \frac{1}{\pi_i} \right\rangle_{\log_{1/q}}; \quad (48)$$

$$S_S(\Pi) = \log \Gamma(\Pi, 1) = \log \left\langle \frac{1}{\pi_i} \right\rangle_{\log}. \quad (49)$$

Eqs. (46) to (49) represent the same entropy measures (35) to (38) after the transformation (42). One does accordingly speak of a *p-picture* and a *π -picture statistics*. This makes it clear how a dual description between a $q > 1$ and a $q < 1$ statistics exists. By means of the KNG-representation, and once identity (45) is known, the TMP-duality between the *p-picture* and the *π -picture statistics* follows immediately without need for further proof.

In the frame of the KNG-representation the TMP-duality reveals to be a "measure form invariant" transformation, i.e. it maintains the q -logarithmic mean over the inverse PDs. A different duality than that of (42) would not lead to same invariance. For instance another duality that has received some attention in the literature is

$$\psi : q \leftrightarrow 2 - q. \quad (50)$$

One can accordingly speak of a *ψ -picture* in this case. We might ask therefore if also this transformation shares similar analytic properties. However, in this case there seems to be no way to define a correspondent dual redefinition of the PD in the *ψ -picture* which describes a normalized distribution and that at the same time leads to form-invariance of the measures, as it is in the case with the TMP-duality. In fact this would imply that, in the same fashion of (45), there must exist a PD $\Psi = \{\psi_1, \dots, \psi_\Omega\}$ such that

$$\Gamma(P, q) = \left\langle \frac{1}{p_i} \right\rangle_{\log_q} = \left(\sum_i p_i^q \right)^{\frac{1}{1-q}} = \left(\sum_i \psi_i^{2-q} \right)^{\frac{1}{q-1}} = \left\langle \frac{1}{\psi_i} \right\rangle_{\log_{2-q}}, \quad (51)$$

and satisfying the normalization condition so that it must have a structure like

$$\psi_i = \frac{g_i(p_i, q)}{\sum_j g_j(p_j, q)}, \quad (52)$$

with g_i some map that transforms smoothly every PD p_i of the *p-picture* into every ψ_i of the *ψ -picture* and such that for $q \rightarrow 1$ it goes like $g_i(p_i, q) \rightarrow p_i$. Inserted in the second r.h.s. term of (51) this means that we are searching for a $g_i(p_i, q)$ that satisfies to

$$\frac{(\sum_i g_i(p_i, q))^{2-q}}{\sum_j g_j(p_j, q)^{2-q}} = \sum_k p_k^q. \quad (53)$$

This seems not to have a simple general solution.

The TMP-duality is special in the sense that, given a normalized PD, it preserves the associated generalized information-entropy measures KNG-representation form invariant.

Note that in Tsallis' measure (47) one transforms the deformation parameter $q \rightarrow \frac{1}{q}$ of Γ , but not the non-additive parameter $r = q$. This makes sense also because otherwise the entropy scalar would not be invariant ($\log_q \Gamma(P, q) = \log_q \Gamma(\Pi, \frac{1}{q}) \neq \log_{\frac{1}{q}} \Gamma(\Pi, \frac{1}{q})$) and the entropy in the π -picture would no longer represent the measure of canonical phase-space volume, against the statistical definition of entropy itself. The two q -parameters of Tsallis' entropy (4) should be handled separately. This might suggest to distinguish between a pseudo-additive and non-extensive parameter. Indeed there is a subtle but potentially important distinction between these two notions which is frequently overlooked (see e.g. [27] and [28]). A single parameter statistics which tries to intertwine them together might lead to an informational loss.

3.2 The TMP-duality as a non-extensive Kramers-Wannier duality

There is a known relation which connects the partition function with the Γ -space of a stationary system. One can generalize to an arbitrary $(\hat{\varepsilon}, q')$ -picture setting ($\hat{\varepsilon} = \hat{p}$, $q' = q$) for the p -picture, ($\hat{\varepsilon} = \hat{\pi}$, $q' = \frac{1}{q}$) for the π -picture, and label \hat{P} and $\hat{\Pi}$ the corresponding MEPD's family. Then, writing the mean energies as $U_g^p = \langle H \rangle^p$ and $U_g^\pi = \langle H \rangle^\pi = \langle H \rangle$, we have

$$Z_{SM}(\hat{\varepsilon}) = \sum_j e_{q'} [-\beta_g^\varepsilon (E_j - U_g^\varepsilon)] = \left(\sum_i (\hat{\varepsilon}_i^{SM})^{q'} \right)^{\frac{1}{1-q'}} = \left\langle \frac{1}{\hat{\varepsilon}_i^{SM}} \right\rangle_{\log_{q'}} = \Gamma(\hat{\varepsilon}, q'). \quad (54)$$

Because, from (40) we have

$$(\hat{\varepsilon}_i^{SM})^{1-q'} (Z_{SM}(\hat{\varepsilon}))^{1-q'} = (e_{q'} [-\beta_g^\varepsilon (E_i - U_g^\varepsilon)])^{1-q'} = 1 - (1 - q') \beta_g^\varepsilon (E_i - U_g^\varepsilon).$$

Then multiplying both members with $(\hat{\varepsilon}_i^{SM})^{q'}$ and summing up, we have

$$\begin{aligned} \sum_i \hat{\varepsilon}_i^{SM} (Z_{SM}(\hat{\varepsilon}))^{1-q'} &= (Z_{SM}(\hat{\varepsilon}))^{1-q'} = \\ &= \sum_i (\hat{\varepsilon}_i^{SM})^{q'} - (1 - q') \beta_g^\varepsilon \sum_i (\hat{\varepsilon}_i^{SM})^{q'} (E_i - U_g^\varepsilon) = \sum_i (\hat{\varepsilon}_i^{SM})^{q'}, \end{aligned}$$

because of the normalization condition and where in the last passage the constraint (39) has been imposed. From this one obtains (54).

And since $Z_{SM}(\hat{P}) = \Gamma(\hat{P}, q)$ and $Z_{SM}(\hat{\Pi}) = \Gamma(\hat{\Pi}, \frac{1}{q})$, then because of (45), what the TMP-duality really implies is that

$$Z_{SM}(\hat{P}) = Z_{SM}(\hat{\Pi}). \quad (55)$$

The partition function represents something which is "picture invariant"⁶.

On the other side, always because of (41), this means also that

$$\sum_j e_q [-\bar{\beta}_g^p E_j] = \sum_j e_{1/q} [-\bar{\beta}_g^\pi E_j], \quad (56)$$

which in general can't hold for the same generalized Lagrangian parameters (i.e. for $\bar{\beta}_g^p = \bar{\beta}_g^\pi$ and $q \neq 1$). The TMP-duality establishes therefore a correspondence between the partition functions of two equilibrated systems with different PDs determined by a Hamiltonian with a $q > 1$ and a $q' = \frac{1}{q} < 1$ statistics and by the Lagrangian parameter $\bar{\beta}_g^p$ and $\bar{\beta}_g^\pi$ respectively.

⁶One has not to accept SM-entropy as a measure for thermostatics to reach the same conclusion because this holds also for Tsallis entropy once we identify (34) with a q -generalized phase-space volume. In the present context we have shown however that this is a more general feature of the formal structure which stands behind Renyi and SM-entropies too.

This equality between the generalized partition functions at two different temperatures⁷ is the q -generalized analogue of the Kramers-Wannier two-dimensional Ising model duality [30] between an ordered and disordered phase (see also [31], and for a more modern introduction [32]). In this scenario a connection is established between a randomly ordered system (typically a non magnetized ferromagnetic substance) above the critical Curie temperature with another system at less than the critical temperature undergoing a symmetry breaking (typically represented by the formation of domain walls).

There have been some quite interesting attempts to expand the Ising model to non-extensive regimes. It can be taken as a good starting point to investigate non-extensivity of systems with long-range interactions presenting critical properties (for some pioneering works in this field see e.g. [29], [33], [34], [35], [36], [37], [38], [39], [40], and others listed in [1]).

More precisely, one can extend the Ising ferromagnet to a long-range interaction Hamiltonian (without magnetic interactions) as

$$H\{\sigma_i\} = - \sum_{\langle ij \rangle} J(J_0, \alpha) \sigma_i \sigma_j \quad (\sigma_i = \pm 1 \forall i) \quad (57)$$

with

$$J(J_0, \alpha) = \frac{J_0}{r_{ij}^\alpha} \quad (J_0 > 0; \alpha > 0), \quad (58)$$

where r_{ij} is the distance (in crystal units) between the i -th and j -th site, $\sum_{\langle ij \rangle}$ is a sum running over all distinct sites of pairs, J_0 is the exchange coupling constant and α determines the power law of the long-range interaction. For $\alpha \rightarrow \infty$ one recovers the first-neighbor model, while for $\alpha \rightarrow 0$ (and rescaling $J_0 \rightarrow J_0/N$) the Curie-Weiss model.

The overall picture which emerges from the above mentioned investigations is that this long-range spin model indeed exhibits non-extensive scaling laws and phase transitions at critical temperatures which depend on α and the space dimensionality d of the system. In particular it turns out that as long as $\alpha > d$ the system is extensive, whereas for $0 \leq \alpha \leq d$ the behavior is non-extensive. The q -parameter seems to depend exclusively on the dimensionality and the long-range interaction power law, i.e. $q \equiv q(\alpha, d)$.

Therefore for the 2d-Ising model the partition function in the $(\varepsilon, q'(\alpha^\varepsilon))$ -picture can be written

$$Z_{SM}(\varepsilon) = \sum_{\sigma_1} \sum_{\sigma_2} \dots \sum_{\sigma_N} e_{q(\alpha^\varepsilon)}[-\bar{\beta}_g^\varepsilon H\{\sigma_i\}], \quad (59)$$

with α^ε the power-law parameter for the chosen picture.

The TMP-duality implies

$$Z_{SM}(\bar{\beta}_g^p, \alpha^p) = Z_{SM}(\bar{\beta}_g^\pi, \alpha^\pi), \quad (60)$$

i.e. it dually relates a system at a temperature T^p governed by a long-range interaction power law $\propto 1/r^{\alpha^p}$ and another system at a temperature T^π with interaction law $\propto 1/r^{\alpha^\pi}$, and with α^π subjected to the condition that $q(\alpha^\pi) = \frac{1}{q(\alpha^p)}$.

3.3 Some remarks on the conceptual foundations of the TMP-duality

The probabilities transformation $p_i \rightarrow \pi_i$ in section (3.1) can be seen also as a "coordinate" transformation. Here one might compare the π -transformation which leads to the KNG-representation form invariance as a statistical analogue of the Lorentz coordinate transformations in special relativity. This analogy holds however only partially since the former is a discrete and the latter a continuous transformation. The TMP-duality does not represent a continuous physical transformation, like a "boost" in special relativity or, more specifically, an isentropic (adiabatic and reversible) process. Moreover, in section (3.2) it links dually two systems at two different temperatures, but these are also governed by different interaction forces among their units. Therefore this kind of statistical duality resembles more the strong-weak coupling duality, acting not only on the coupling constants but also on the power laws of the long-range forces determining the non-extensive behavior of the system. It relates two problems with different couplings, mapping the strong coupling regime into the weak one. Dualities seem to be

⁷What kind of relation exists between the generalized non-extensive Lagrangian parameter and the ordinary statistical notion of "temperature" is actually a matter of debate, and what kind of non-trivial rescaling of the system's temperatures in the different pictures must be employed is therefore not entirely clear (see however a paper by Salazar and Toral [29] in this sense), but this is at present not so decisive for the present work and we will generically speak of a system at temperature T^p and T^π for the respective pictures.

an underlying common feature of physical theories in general, as for instance in the case of the electromagnetic duality. Especially in modern quantum gravity theories, as in superstring theories, these kind of dualities turned out to be useful in describing the theory from a nonperturbative standpoint making calculations easier and uniting apparently different theories in one comprehensive framework. The duality described here, i.e. that which leads to eq. (60), might potentially also represent a tool to simplify the calculations of the thermodynamic properties of a complex non-extensive system governed by non-trivial long-range interaction force laws studying its (eventually extensive) dual counterpart with differently scaled interaction forces. A more precise understanding of the conceptual foundations is of course desirable.

4 Conclusion

Generalized information-entropy measures have been presented in the light of a formalism based in particular on the manipulation of the generalized exponential and logarithm functions together with Kolmogorov's and Nagumo's averaging. This kind of formal approach to a generalized statistics seems to lend itself particularly well in describing the deeper implications of the $q \leftrightarrow \frac{1}{q}$ duality of a q -generalized statistics which goes beyond that described by Tsallis entropy functionals. We showed how this duality implies a KNG-representation entropy form invariance. The $q \rightarrow 2 - q$ duality seems to be less fundamental in this sense: unlike the TMP-duality it does not preserve a KNG-representation form invariance for a normalized PD. The TMP-duality establishes also an equivalence between two partition functions for different Lagrangian parameters and long-range interaction power laws in the p and π -pictures respectively. A possible physical meaning of this is that it can be interpreted as a 2d Ising model duality generalized to non-extensive systems.

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